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A stable well-conditioned integral equation for electromagnetism scattering

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Abstract

Finding a formulation for electromagnetic scattering of surfaces which is both well-posed and produces a well-conditioned linear system is still a challenging problem. We here propose one such formulation valid in the high-frequency regime. The mathematical analysis is provided and numerical results on rather complex geometries show the performance of the method.

Keywords: Time-harmonic scattering problem; Maxwell equations; Iterative method; Integral equation; Helmholtz decomposition; CFIE; Preconditioner

1. Introduction

The use of integral equations to solve wave scattering has become very popular from the sixties since the method intrinsically permits to reduce by one the dimension of the problem (e.g., to pose the problem on a scattering surface only instead of the whole space). However, it appeared that several formulations, interesting at first sight because of their simplicity of discretization or their physical meaning, are actually ill-posed for some (bad) frequencies. Adapting to acoustics ideas that were developed in the context of elasticity by the Russian school (mainly represented by Kupradze [14]), Brackhage and Werner [6] and Panich [22] observed that a linear combination of equations which do not possess the same irregular frequencies may (provided the coupling coefficients are well chosen) produce a non-resonant equation. When applied to electromagnetism, this technique will be very fruitful to write well-posed equations at all frequencies. Maybe the two most famous ones are those developed by Mitzner [21] which is a combined field integral equation (CFIE) and the one by Mautz and Harrington [19] which is a combined source integral equations (CSIE). Although both equations are based on common principles, the CFIE will be the most successful. Easier to implement, and more natural from a physical viewpoint (maybe also more precise than its counterpart), the CFIE belongs now to the family of classical equations, and most of the industrial codes devoted to electromagnetism solve it.

Nonetheless, although stable equations exist, when people turned to solve bigger and bigger cases (obtained by finer and finer discretization), the use of direct solvers became impossible, and the switching to iterative methods (the development of GMRES [25] brought a substantial improvement in that direction) posed both the question of finding a

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fast matrix–vector multiplication, and the question of speeding up the convergence of the iterative method. The former problem is now considered to be solved by using the fast multipole method (FMM) of Rokhlin (see [24] and later on the papers by Greengard and Rokhlin [12] for instance) which reduces the $O(N^2)$ classical algorithm for matrix–vector multiplication to a $O(N\log(N))$ complexity for such problems, and allows now, on modern computers, to consider problems with a number of unknowns exceeding the million.

The remaining question is those of convergence speed of the iterative method (GMRES for instance) to solve the linear system coming from the discretization of the problem. It is well-known that it is directly related to the condition number of the matrix, and it turned out that though well-posed for all frequencies the best formulations (like the CFIE) may produce a very ill-conditioned linear system for which the numerical resolution by means of iterative methods becomes costly. The traditional cure for this problem is to use a preconditioner to the linear system, which is roughly equivalent to multiply the original system by a matrix such that the resulting matrix is close to identity. However, such preconditioners are not easy to compute as their pure algebraic nature may not easily take into account the dependence in the frequency. The efficiency of these preconditioners is moreover difficult to analyze theoretically and sometimes also not quite convincing from a practical point of view.

The following idea (though for different problems [20,26,27]) was to stabilize the formulation not after their discretization (after the assembling of the linear system), but at the very beginning of the conception of the integral equation, by finding a parametrix of the underlying operator. We take this point of view here in the framework of scattering. As it has been pointed out by several authors [8,10], though conceptually clear, this program is not easy to concretize in practice. This paper is devoted to give an example of such a strategy. Indeed, Section 3 shows how to construct a new intrinsically well-conditioned integral equation which leads after discretization to linear systems for which classical iterative methods converge quickly without the need of any preconditioner. Section 2 makes a (small) review of integral equations in sources versus in fields, where it is shown that for our objectives the source formulations may be more suitable. The discretization is treated in Section 4 and numerical results demonstrating the robustness and real applicability of the method are provided in Section 5.

2. Integral equations in sources and in fields

In this section, we recall the model problem on which we work and the classical integral equations usually used to solve it. We will focus on two different strategies: the CSIEs and the CFIEs.

Let Ω be a three-dimensional bounded domain with a smooth boundary Γ. We call $\mathbf{n}$ the outward unit normal on Γ. We define W^+ as the space of radiating electric fields $\mathbf{E}$ solutions of Maxwell equations in $\mathbb{R}^3\backslash\overline{\Omega}$ which have a tangential trace on Γ. Our problem writes as follows:

$$\text{Find } \mathbf{E}\in W^+ \text{ such that } \mathbf{n}\times\mathbf{E}=-\mathbf{n}\times\mathbf{E}^{\mathrm{inc}} \quad \text{on } \Gamma, \tag{1}$$

which models the scattering by a perfectly conducting material of an incident wave $\mathbf{E}^{\mathrm{inc}}$.

A rather natural way to solve (1) with an integral equation, consists in giving us a parameterization of admissible fields W^+ with a functional which links current distributions on Γ and electric fields of W^+

$$\mathscr{V}:\mathscr{D}'_T(\Gamma)\to W^+, \tag{2}$$

where $\mathscr{D}'_T(\Gamma)$ is the space of tangential vectorial distributions on Γ. The corresponding integral equation becomes

$$\mathbf{n}\times\mathscr{V}(\mathbf{u})=-\mathbf{n}\times\mathbf{E}^{\mathrm{inc}} \quad \text{on } \Gamma, \tag{3}$$

where the unknown $\mathbf{u}$ is not necessarily physically meaningful (in other words, $\mathbf{u}$ does not need to be a Cauchy data of the solution $\mathbf{E}$ to (1)). Classical potentials in $\mathbb{R}^3\backslash\Gamma$ are given by

$$\mathscr{L}=\frac{1}{\mathrm{i}k}\nabla\times\nabla\times\mathscr{G} \quad \text{and} \quad \mathscr{K}=\nabla\times\mathscr{G}, \tag{4}$$

where $\mathscr{G}$ stands for the vector potential (which depends on the wavenumber k) which to a tangent vectorfield $\mathbf{u}$ on Γ associates the vector-field defined on $\mathbb{R}^3\backslash\Gamma$ by

$$\mathscr{G}\mathbf{u}(x)=-\frac{1}{4\pi}\int_\Gamma\frac{\mathrm{e}^{\mathrm{i}k|x-y|}}{|x-y|}\mathbf{u}(y)\,\mathrm{d}y. \tag{5}$$

However, it may happen that equations built from (3) are ill-posed (for some irregular frequencies for instance). A way of circumventing this problem consists in combining the potentials $\mathscr{L}$ and $\mathscr{K}$. For instance, Mautz and Harrington proposed in [19] to use $\mathscr{V}(\mathbf{u}) = \mathscr{L}(\mathbf{u}) - \mathscr{K}(\alpha\mathbf{n} \times \mathbf{u})$. In the same spirit, it is tempting to use the potential

$$\mathscr{V}(\mathbf{u}) = \mathscr{L}(\alpha\,\mathbf{n} \times \mathbf{u}) - \mathscr{K}(\mathbf{u}), \tag{6}$$

which leads to the CSIE

$$\mathbf{n} \times L(\alpha\mathbf{n} \times \mathbf{u}) - \mathbf{n} \times K(\mathbf{u}) = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} \quad \text{on } \Gamma, \tag{7}$$

where $L\mathbf{u}$ and $K\mathbf{u}$ stand for the exterior tangential trace on Γ of $\mathscr{L}\mathbf{u}$ and $\mathscr{K}\mathbf{u}$, respectively. In [21,4], it is shown that both formulations are well-posed at all frequencies provided $Re(\alpha) > 0$. In Section 3 we generalize this writing.

Although this approach is very natural from a theoretical viewpoint, people have preferred to use in applications another kind of integral equations in which the unknown is the Cauchy data of the problem. The starting point of these formulations is probably the Stratton–Chu formulas which allow to reconstruct completely an electromagnetic field $(\mathbf{E}, \mathbf{H})$ of W^+ from its tangential traces:

$$\mathbf{E} = \mathscr{L}(\mathbf{n} \times \mathbf{H}) - \mathscr{K}(\mathbf{n} \times \mathbf{E}), \quad \mathbf{H} = -\mathscr{K}(\mathbf{n} \times \mathbf{H}) - \mathscr{L}(\mathbf{n} \times \mathbf{E}). \tag{8}$$

As $\mathbf{E}^{\mathrm{inc}}$ does not satisfy the radiating condition, $\mathbf{E}^{\mathrm{inc}} \notin W^+$. Therefore, formulas (8) do not apply and we instead have [9]

$$0 = \mathscr{L}(\mathbf{n} \times \mathbf{H}^{\mathrm{inc}}) - \mathscr{K}(\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}), \quad 0 = -\mathscr{K}(\mathbf{n} \times \mathbf{H}^{\mathrm{inc}}) - \mathscr{L}(\mathbf{n} \times \mathbf{E}^{\mathrm{inc}}). \tag{9}$$

Summing up (8) and (9) and using that the material is perfectly conducting leads to the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE), respectively:

$$L(\mathbf{J}) = -\mathbf{E}_T^{\mathrm{inc}} \text{ (EFIE)}, \quad \mathbf{n} \times K(\mathbf{J}) + \mathbf{J} = \mathbf{n} \times \mathbf{H}^{\mathrm{inc}} \text{ (MFIE)}, \tag{10}$$

where the unknown $\mathbf{J}$ is equal to $\mathbf{n} \times (\mathbf{H} + \mathbf{H}^{\mathrm{inc}})$, and $\mathbf{E}_T^{\mathrm{inc}}$ is the tangential component of $\mathbf{E}^{\mathrm{inc}}$.

To avoid the appearance of irregular frequencies, Mitzner [21] proposed to combine both field equations to get the so-called CFIE which turns out to be well-posed at every frequency:

$$\alpha L(\mathbf{J}) + \mathbf{n} \times K(\mathbf{J}) + \mathbf{J} = -\alpha\mathbf{E}_T^{\mathrm{inc}} + \mathbf{n} \times \mathbf{H}^{\mathrm{inc}} \quad \text{(CFIE)}. \tag{11}$$

This latter approach in fields turned out to be the most popular, probably because the unknowns are physically relevant quantities. Bendali in [3] claims for instance that "the most used methods and the most interesting ones are those whose unknowns have a clear physical meaning". However, using only physical unknowns is very constraining. Source equations in turn, are richer, and give tools to build formulations that are more stable than field equations. An example of such formulation is given in Section 3 which is very stable, even at high frequencies.

3. Generalization of a CSIE using an admittance operator

3.1. GCSIE formalism

The formalism we present follows ideas developed by one of the author in [17] though for acoustics. It is also not restricted to wave propagation problems as explained in [18] where it is shown that such a strategy may be applied to any problem governed by a linear elliptic PDE for which a Green kernel can be computed numerically. Moreover, boundary conditions more sophisticated than those considered here (perfectly conducting material) may also be treated.

The governing idea is to find a formulation which is intrinsically well conditioned. The key observation in our new formalism is that Stratton–Chu formulas (8) may be written in terms of only one of the two tangential traces $\mathbf{n} \times \mathbf{E}$ or $\mathbf{n} \times \mathbf{H}$ only. Indeed, since the original problem (1) possesses a unique solution, both quantities are linked by an operator Y^+:

$$\mathbf{n} \times \mathbf{H} = Y^+(\mathbf{n} \times \mathbf{E}) \quad \text{on } \Gamma. \tag{12}$$

Mathematicians call Y^+ the Dirichlet-to-Neumann or Steklov–Poincaré operator whereas physicists prefer to use the term of admittance. With the help of Y^+, one may write any $\mathbf{E} \in W^+$ as

$$\mathbf{E} = \mathscr{L}Y^+(\mathbf{n} \times \mathbf{E}) - \mathscr{K}(\mathbf{n} \times \mathbf{E}), \tag{13}$$

from which one deduces by taking the trace on Γ

$$\mathbf{n} \times \mathbf{E} = \mathbf{n} \times LY^+(\mathbf{n} \times \mathbf{E}) - \mathbf{n} \times K(\mathbf{n} \times \mathbf{E}). \tag{14}$$

Hence, if one wants to solve (1) with a source equation, the best possible potential is given by $\mathscr{V} = \mathscr{L}Y^+ - \mathscr{K}$, which leads from (14) to $\mathbf{n} \times \mathscr{V} = \mathrm{Id}$, from which the solution to (3) is immediately given by the right-hand side. Of course, the drawback of the method is that, unless on very special geometries, the operator Y^+ is never known explicitly. The idea consists in finding a numerical approximation $\widetilde{Y}^+$ of Y^+, and using $\mathscr{V} = \mathscr{L}\widetilde{Y}^+ - \mathscr{K}$. In this way, we get the generalized CSIE:

$$\mathbf{n} \times L\widetilde{Y}^+\mathbf{u} - \mathbf{n} \times K\mathbf{u} = -\mathbf{n} \times \mathbf{E}^{\mathrm{inc}} \ \text{(GCSIE)}. \tag{15}$$

Indeed, taking $\widetilde{Y}^+ = \alpha\mathbf{n}\times$ leads back to the CSIE of Mautz and Harrington [19]. However, taking $\widetilde{Y}^+ = Y^+$ yields an underlying operator equal to the identity. It is, therefore, reasonable to conjecture that the closer $\widetilde{Y}^+$ to Y^+ (in a sense to be precised), the better (15) is conditioned.

3.2. A new well-posed equation for high frequencies

The GCSIE formalism is very general and depends upon the choice of an operator which is expected to approximate the admittance operator. Several possibilities exist depending on the application. For instance one might construct different approximation for the low- or high-frequency regime. We hereafter propose an approximation particularly dedicated to the high-frequency situation which relies on a localization of the admittance Y^+ similarly to the idea used in [15] for acoustics. Let us quote that in the case of acoustics, another approach, based on a microlocal approximation of the Dirichlet-to-Neumann operator has been proposed in [1,11]. The results are encouraging but the technique has not been yet applied to electromagnetics.

It is well known that in the high-frequency regime, scattering phenomena tend to localize (e.g., asymptotic theories prove that the energy at high frequency localize near the rays given by geometrical optics). We, therefore, conjecture that the admittance operator itself tends to localize at high frequency, and $-2\,\mathbf{n} \times L$ being the admittance of an infinite metallic plate (see [5] for a precise analysis), we propose to set

$$\widetilde{Y}^+ = -2\sum_p \chi_p \mathbf{n} \times L\chi_p. \tag{16}$$

In this equation, we have taken $(U_p, \chi_p)_p$ to be a quadratic partition of Γ, which means that $(U_p)_p$ is a family of open set included in Γ such that $\cup_p U_p = \Gamma$ and $(\chi_p)_p$ is a family of smooth functions such that the support of χ_p is included in U_p for all p and

$$\forall x \in \Gamma, \quad \sum_p \chi_p^2(x) = 1. \tag{17}$$

We stress on the fact that each U_p must be an open surface, and cannot cover the whole boundary Γ. It is particularly important condition in order to prove the Proposition 5. In what follows, we prove that under a natural condition on χ_p, the equation obtained by taking (16) is well posed for large enough frequencies. The strategy is classical, and consists in using Fredholm's theory. We first demonstrate that the GCSIEs operator is a compact perturbation of a coercive order 0 operator, and then that it is one-to-one.

We recall that W^+ is the space of radiating electric fields $\mathbf{E}$ solutions of Maxwell equations in $\mathbb{R}^3\backslash\overline{\Omega}$ and having a tangential trace on Γ. We also denote by W^- the space of electric fields $\mathbf{E}$ solutions of Maxwell equations in Ω and having a tangential trace on Γ.

Let $H^s_T(\Gamma)$ be the usual Sobolev space of tangential fields of order $s \in \mathbb{R}$, with the norm $|\cdot|_s$.

We denote by X^+ (resp. X^-) the space of the trace couples $(\sigma_0^+\mathbf{E}, \sigma_1^+\mathbf{E}) \in \mathscr{D}_T(\Gamma) \times \mathscr{D}_T(\Gamma)$ (resp. $(\sigma_0^-\mathbf{E}, \sigma_1^-\mathbf{E})$) where $\mathbf{E} \in W^+$ (resp. $\mathbf{E} \in W^-$), and where $\sigma_0^\pm = \mathbf{n} \times \gamma_T^\pm$ and $\sigma_1^\pm = (1/\mathrm{i}k)\mathbf{n} \times \gamma_T^\pm \nabla\times$. We have denoted by γ_T^+ (resp. γ_T^-) the tangential outer (resp. inner) trace.

We define the operators

$$C^+ = \sigma_0^+ \oplus \sigma_1^+(\mathscr{C}^+), \quad C^- = \sigma_0^- \oplus \sigma_1^-(\mathscr{C}^-),$$

where the potentials $\mathscr{C}^+$ and $\mathscr{C}^-$ are given by

$$\begin{aligned} &\mathscr{C}^+ : \mathscr{D}'_T(\Gamma) \times \mathscr{D}'_T(\Gamma) \to W^+, \quad &&\mathscr{C}^- : \mathscr{D}'_T(\Gamma) \times \mathscr{D}'_T(\Gamma) \to W^-. \\ &(\mathbf{f}, \mathbf{g}) \to \mathscr{L}\mathbf{g} - \mathscr{K}\mathbf{f} \quad &&(\mathbf{f}, \mathbf{g}) \to -\mathscr{L}\mathbf{g} + \mathscr{K}\mathbf{f} \end{aligned}$$

The operators C^+ and C^- are the so-called *Calderón projectors* [7] which decompose the space $X = \mathscr{D}_T(\Gamma) \times \mathscr{D}_T(\Gamma)$ into the direct sum $X = X^+ \oplus X^-$. For the potential $\mathscr{L}$, this decomposition gives the continuity and jump relations

$$\sigma_0^+\mathscr{L}\mathbf{u} = \sigma_0^-\mathscr{L}\mathbf{u} = \mathbf{n} \times L\mathbf{u} \quad \forall \mathbf{u} \in \mathscr{D}_T(\Gamma), \tag{18}$$

$$\sigma_1^+\mathscr{L}\mathbf{u} - \sigma_1^-\mathscr{L}\mathbf{u} = \mathbf{u} \quad \forall \mathbf{u} \in \mathscr{D}_T(\Gamma). \tag{19}$$

The crucial result that enables us to prove that the GCSIEs operator is a compact perturbation of a coercive order 0 operator, is the (orthogonal) *Helmholtz decomposition* $H_T^0(\Gamma) = \mathbf{n} \times \nabla(H^1(\Gamma)) \oplus \nabla(H^1(\Gamma))$, whose associated projectors are, respectively, $\Pi_{\mathrm{loop}} = -\mathbf{n} \times \nabla\Delta^{-1}\nabla \cdot \mathbf{n}\times$ and $\Pi_{\mathrm{star}} = \nabla\Delta^{-1}\nabla\cdot$, where Δ^{-1} is the pseudo-inverse of the Laplace–Beltrami operator (which is defined by first projecting on null average functions, and then applying Δ^{-1}). We refer to [16] for more details. In the sequel, we will call $G = \gamma_T^+\mathscr{G} = \gamma_T^-\mathscr{G}$ the tangential trace of the potential $\mathscr{G}$, and G_s the Dirichlet trace of the single layer potential of acoustics (the scalar convolution with Helmholtz's Green kernel). We will use the classical writing $L = -\mathrm{i}kG - (\mathrm{i}/k)\nabla G_s\nabla\cdot$. We stress on the fact that the boundary Γ has been assumed to be smooth allowing us to use the pseudodifferential calculus on Γ. Therefore, we call $\Psi^s(\Gamma)$ the class of pseudo-differential operators of order s defined on Γ. It is known that principal symbols of G_s and G are, respectively, $-1/2\|\xi\|$ and $-1/2\|\xi\|\mathrm{Id}$, and since Δ^{-1} is pseudo-differential with principal symbol $-\|\xi\|^{-2}$, it is, therefore, equivalent to$-4G_s^2$ modulo $\Psi^{-3}(\Gamma)$. In what follows, we will frequently use the notation:

$$\mathbf{\Lambda}_n = \nabla_\Gamma \chi_n. \tag{20}$$

Lemma 1. *Denoting by* $[A, B] = AB - BA$, *the commutator of A and B*, *we have*

$$[G_s, \chi_n] \sim -G_s(\Lambda_n \cdot \nabla\Delta^{-1}) \bmod \Psi^{-3}(\Gamma), \tag{21}$$

$$[G, \chi_n] \in \Psi^{-2}(\Gamma), \tag{22}$$

$$G\mathbf{n} \times \nabla - \mathbf{n} \times \nabla G_s \in \Psi^{-1}(\Gamma). \tag{23}$$

Proof. One can easily see [4] that the principal symbol of $[G_s, \chi_n]$ is $\sigma(x, \xi) = (1/2\mathrm{i}\|\xi\|^3)\xi \cdot \mathbf{\Lambda}_n(x)$. Then (21) derives from the fact that $[G_s, \chi_n]$ and $-G_s(\mathbf{\Lambda}_n \cdot \nabla\Delta^{-1})$ have the same principal symbol of order -2. Eqs. (22) and (23) immediately follow from the diagonal structure of the principal symbol of G. □

The projectors of the Helmholtz decomposition Π_{loop} and Π_{star} together with Lemma 1 enable us to clarify the nature of the operator $\mathbf{n} \times L\widetilde{Y}^+$.

Proposition 2. *The operator* $\mathbf{n} \times L\widetilde{Y}^+$ *is of order 0 and we have*

$$\mathbf{n} \times L\widetilde{Y}^+ \sim \frac{1}{2}\,\mathrm{Id} - \frac{1}{2k^2}\,\Pi_{\mathrm{loop}} \sum_n \mathbf{\Lambda}_n\,\mathbf{\Lambda}_n \cdot \Pi_{\mathrm{loop}} \bmod \Psi^{-1}(\Gamma). \tag{24}$$

Proof. We start by making a few symbolic calculations. Namely, since the operator G is of order -1, we have

$$\mathbf{n} \times G\left(\sum_n \chi_n\mathbf{n} \times G\chi_n\right) \in \Psi^{-2}(\Gamma). \tag{25}$$

Next, from $G_s^2 \sim -\Delta^{-1}/4$, mod $\Psi^{-3}(\Gamma)$, $\sum_n \chi_n^2 = 1$, (22) and (23), we get modulo $\Psi^{-1}(\Gamma)$

$$\mathbf{n} \times G\left(\sum_n \chi_n \mathbf{n} \times \nabla G_s \nabla \cdot \chi_n\right) \sim -\sum_n \chi_n \nabla G_s^2 \nabla \cdot \chi_n \sim \frac{1}{4}\Pi_{\text{star}}, \tag{26}$$

$$\mathbf{n} \times \nabla G_s \nabla \cdot \left(\sum_n \chi_n \mathbf{n} \times G\chi_n\right) \sim \mathbf{n} \times \nabla G_s^2 \nabla \cdot \mathbf{n} \times \sim \frac{1}{4}\Pi_{\text{loop}}. \tag{27}$$

Eventually, we set $A = \mathbf{n} \times \nabla G_s \nabla \cdot (\sum_n \chi_n \mathbf{n} \times \nabla G_s \nabla \cdot \chi_n)$. The operator A is a priori of order 2, as a product of two order 1 operators, but it is easily seen that it is actually of order 0. Indeed,

$$\begin{aligned} A &= \mathbf{n} \times \nabla G_s \nabla \cdot \left[\mathbf{n} \times \nabla\left(\sum_n \chi_n G_s \nabla \cdot \chi_n\right) - \sum_n (\mathbf{n} \times \mathbf{\Lambda}_n) G_s \nabla \cdot \chi_n\right] \\ &= -\mathbf{n} \times \nabla G_s \nabla \cdot \sum_n (\mathbf{n} \times \mathbf{\Lambda}_n) G_s \nabla \cdot \chi_n \end{aligned}$$

thanks to $\nabla \cdot (\mathbf{n} \times \nabla) = 0$. Developing the term $\nabla \cdot \chi_n$, we obtain

$$\begin{aligned} A &= -\mathbf{n} \times \nabla G_s \nabla \cdot \sum_n (\mathbf{n} \times \mathbf{\Lambda}_n) G_s (\mathbf{\Lambda}_n \cdot + \chi_n \nabla \cdot) \\ &= -\mathbf{n} \times \nabla G_s \nabla \cdot \sum_n (\mathbf{n} \times \mathbf{\Lambda}_n)(G_s \mathbf{\Lambda}_n \cdot + \chi_n G_s \nabla \cdot + [G_s, \chi_n]\nabla \cdot). \end{aligned}$$

Since the cut-off functions $(\chi_n)_n$ form a quadratic partition of unity, we have $\sum_n \chi_n \mathbf{\Lambda}_n = 0$ and therefore,

$$A = -\mathbf{n} \times \nabla G_s \nabla \cdot \sum_n (\mathbf{n} \times \mathbf{\Lambda}_n)(G_s \mathbf{\Lambda}_n \cdot + [G_s, \chi_n]\nabla \cdot),$$

which is at most of order 0 since $[G_s, \chi_n]$ is of order -2.

Using (21) and $G_s^2 \sim -\Delta^{-1}/4$ mod $\Psi^{-3}(\Gamma)$, we get modulo $\Psi^{-1}(\Gamma)$:

$$-\mathbf{n} \times \nabla G_s \nabla \cdot \left(\sum_n (\mathbf{n} \times \mathbf{\Lambda}_n) G_s \mathbf{\Lambda}_n \cdot\right) \sim -\frac{1}{4}\Pi_{\text{loop}} \sum_n \mathbf{\Lambda}_n \mathbf{\Lambda}_n \cdot, \tag{28}$$

$$-\mathbf{n} \times \nabla G_s \nabla \cdot \sum_n (\mathbf{n} \times \mathbf{\Lambda}_n)[G_s, \chi_n]\nabla \cdot \sim \frac{1}{4}\Pi_{\text{loop}} \sum_n \mathbf{\Lambda}_n \mathbf{\Lambda}_n \cdot \Pi_{\text{star}}. \tag{29}$$

Summing (28) and (29) and using $\Pi_{\text{loop}} + \Pi_{\text{star}} = \mathrm{Id}$, we get

$$A \sim -\frac{1}{4}\Pi_{\text{loop}} \sum_n \mathbf{\Lambda}_n \mathbf{\Lambda}_n \cdot \Pi_{\text{loop}} \text{ mod } \Psi^{-1}(\Gamma). \tag{30}$$

Now, when one computes $\mathbf{n} \times L\widetilde{Y}^+$ from $L = -\mathrm{i}k\, G - (\mathrm{i}/k)\nabla G_s \nabla \cdot$ and the expression of $\widetilde{Y}^+$ is given by (16), one obtains, thanks to symbolic results (25)–(30), the announced equivalence (24). $\square$

Now, calling S the order 0 operator $S = -(1/2k^2)\Pi_{\text{loop}} \sum_n \mathbf{\Lambda}_n \mathbf{\Lambda}_n \cdot \Pi_{\text{loop}}$, and since $\mathbf{n} \times K \sim -\frac{1}{2}\mathrm{Id}$ mod $\Psi^{-1}(\Gamma)$, Proposition 2 enables us to write GCSIEs operator $T = \mathbf{n} \times L\widetilde{Y}^+ - \mathbf{n} \times K$ as $\mathrm{Id} + S + C$, with C compact in $H_T^s(\Gamma)$.

Proposition 3 (*Coerciveness of* $\mathrm{Id} + S$). *Under the condition that*

$$\max_{x \in \Gamma}\left(\sum_n |\mathbf{\Lambda}_n(x)|^2\right) < 2k^2, \tag{31}$$

the operator $\mathrm{Id} + S$ *is coercive in* $H^0_T(\Gamma)$: *there exists* $C > 0$ *such that*

$$\langle (\mathrm{Id} + S)\mathbf{u}, \overline{\mathbf{u}}\rangle \geqslant C|\mathbf{u}|_0^2, \quad \forall \mathbf{u} \in H^0_T(\Gamma).$$

Proof. Let $\mathbf{u} \in H^0_T(\Gamma)$. Then

$$\langle S\mathbf{u}, \overline{\mathbf{u}}\rangle = -\frac{1}{2k^2}\sum_n |\mathbf{\Lambda}_n \cdot \Pi_{\mathrm{loop}}\mathbf{u}|_0^2 \geqslant -\frac{1}{2k^2}\max_{x\in\Gamma}\left(\sum_n |\mathbf{\Lambda}_n(x)|^2\right)|\Pi_{\mathrm{loop}}\mathbf{u}|_0^2,$$

and since $|\Pi_{\mathrm{loop}}\mathbf{u}|_0 \leqslant |\mathbf{u}|_0$, we obtain

$$\langle (\mathrm{Id} + S)\mathbf{u}, \overline{\mathbf{u}}\rangle \geqslant \left[1 - \frac{1}{2k^2}\max_{x\in\Gamma}\left(\sum_n |\mathbf{\Lambda}_n(x)|^2\right)\right]|\mathbf{u}|_0^2. \tag{32}$$

As a compact perturbation of a coercive operator, T is Fredholm of index 0 in $H^0_T(\Gamma)$.

Now we prove that T is one-to-one in $H^0_T(\Gamma)$. We start by recalling a well-known result for interior and exterior traces [9]. □

Lemma 4. *The interior and exterior traces satisfy the following properties*:

(P1) *If* $(\mathbf{u}, \mathbf{v}) \in X^-$, *then* $\mathrm{Re}(\langle \mathbf{v}, \mathbf{n} \times \overline{\mathbf{u}}\rangle) = 0$.
(P2) *If* $(\mathbf{u}, \mathbf{v}) \in X^+\backslash\{(0, 0)\}$, *then* $\mathrm{Re}(\langle \mathbf{v}, \mathbf{n} \times \overline{\mathbf{u}}\rangle) > 0$.

Here, $\langle ., .\rangle$ *denotes the product in* $H^0_T(\Gamma)$ *and* $\overline{\mathbf{u}}$ *the complex conjugate of* $\mathbf{u}$.

Proposition 5. *The coupling operator* $\widetilde{Y}^+$ *satisfies the positivity property*:

$$\textit{If } \mathbf{u} \in \mathscr{D}_T(\Gamma)\backslash\{0\}, \quad \mathrm{Re}(\langle \widetilde{Y}^+\mathbf{u}, \mathbf{n} \times \overline{\mathbf{u}}\rangle) > 0. \tag{33}$$

Proof. Let $\mathbf{E}_n = \mathscr{L}(\chi_n\mathbf{u})$. By (18) and with the jump relation (19),

$$\begin{aligned}\mathrm{Re}(\langle \widetilde{Y}^+\mathbf{u}, \mathbf{n} \times \overline{\mathbf{u}}\rangle) &= -2\sum_n \mathrm{Re}(\langle \mathbf{n} \times L\chi_n\mathbf{u}, \mathbf{n} \times \chi_n\overline{\mathbf{u}}\rangle) \\ &= -2\sum_n \mathrm{Re}(\langle \sigma_0^+\mathbf{E}_n, \mathbf{n} \times \overline{(\sigma_1^+ - \sigma_1^-)\mathbf{E}_n}\rangle).\end{aligned}$$

From (18) and the property (P1), we get:

$$\mathrm{Re}(\langle \widetilde{Y}^+\mathbf{u}, \mathbf{n} \times \overline{\mathbf{u}}\rangle) = -2\sum_n \mathrm{Re}(\langle \sigma_0^+\mathbf{E}_n, \mathbf{n} \times \overline{\sigma_1^+\mathbf{E}_n}\rangle).$$

The property (P2) allows us to conclude that this sum is positive. If the sum vanishes, from (P2), $\sigma_0^+\mathbf{E}_n = 0$ for all n. Since the solution of the exterior Maxwell problem is unique, $\mathbf{E}_n = 0$ in $\mathbb{R}^3\backslash\overline{\Omega}$. Moreover, $\mathbf{E}_n$ is analytic in $\mathbb{R}^3\backslash U_n$. The surface U_n being open, we can conclude that $\mathbf{E}_n = 0$ in $\mathbb{R}^3\backslash U_n$. Then $\sigma_1^+\mathbf{E}_n = \sigma_1^-\mathbf{E}_n = 0$, and, with the jump relation (19), $\chi_n\mathbf{u} = 0$ for all n. Therefore $\mathbf{u} \equiv 0$, which establishes the positivity property (33) of $\widetilde{Y}^+$.

Let $\mathbf{u} \in \mathscr{D}_T(\Gamma)\backslash\{0\}$. By Proposition 5, $\mathrm{Re}(\langle \widetilde{Y}^+\mathbf{u}, \mathbf{n} \times \overline{\mathbf{u}}\rangle) > 0$. The property (P1) implies that $(\mathbf{u}, \widetilde{Y}^+\mathbf{u}) \notin X^-$ and then $(\mathbf{u}, \widetilde{Y}^+\mathbf{u})$ has a component $C^+(\mathbf{u}, \widetilde{Y}^+\mathbf{u}) \neq 0$. Therefore, we immediately have $\sigma_0^+\mathscr{C}^+(\mathbf{u}, \widetilde{Y}^+\mathbf{u}) \neq 0$, and the GCSIEs operator $\sigma_0^+\mathscr{C}^+(\mathrm{Id} \oplus \widetilde{Y}^+)$ is one-to-one in $\mathscr{D}_T(\Gamma)$.

We recall that the GCSIEs operator T can be written as $\mathrm{Id} + S + C$ with C compact in $H^s_T(\Gamma)$. It is clear that under (31) $\mathrm{Id} + S$ is uniformly elliptic and, therefore, hypoelliptic. We deduce that T is hypoelliptic. Being one-to-one in $\mathscr{D}_T(\Gamma)$, it is one-to-one in $H^0_T(\Gamma)$. The Riesz–Fredholm theory predicts that the GCSIE has a unique solution in $H^0_T(\Gamma)$. We have, therefore, proved the theorem. □

Theorem 6. *The problem* (15) *is well-posed on* $H^0_T(\Gamma)$ *under condition* (31).

4. Discretization of the equation

We assume that the surface Γ is replaced by a triangulation, whose thinness is given by the parameter h. We adopt the Raviart–Thomas finite elements of order 0 with H_{div} conformity [23]. The degree of freedom (DoF) associated with the edge i is noted $\mathbf{e}_i$. We apply a Galerkin scheme to the GCSIE:

$$M_h^{-1}([\mathbf{n}\times L\widetilde{Y}^+]_h\mathbf{u}_h - [\mathbf{n}\times K]_h\mathbf{u}_h) = -M_h^{-1}[\mathbf{n}\times\mathbf{E}^{\mathrm{inc}}]_h, \tag{34}$$

where $([A]_h)_{i,j} = \langle A\mathbf{e}_j, \mathbf{e}_i\rangle$, $(M_h)_{i,j} = \langle \mathbf{e}_j, \mathbf{e}_i\rangle$ and $([\mathbf{n}\times\mathbf{E}^{\mathrm{inc}}]_h)_i = \langle \mathbf{n}\times\mathbf{E}^{\mathrm{inc}}, \mathbf{e}_i\rangle$.

The construction of the matrix $[\mathbf{n}\times K]_h$ is not problematic, since it is almost the same as the one built for the MFIE (10). The main difficulty for the discretization of the GCSIE remains the construction of the matrix $[\mathbf{n}\times L\widetilde{Y}^+]_h$. It is not possible to envisage a direct construction due to the product of the pseudo-differential operators $\mathbf{n}\times L$ and $\widetilde{Y}^+$. Nevertheless, we propose to approach this matrix by a product $[\mathbf{n}\times L]_h\widetilde{Y}_h^+$, where $\widetilde{Y}_h^+$ is a linear operator to construct. We quote that the Galerkin matrix $[\mathbf{n}\times L]_h$ is not so easy to build as the more familiar matrix $[L]_h$ of EFIE equation. A term of the form $\nabla_\Gamma\cdot(\mathbf{n}\times\mathbf{u})$ appears, needing to take into account some Dirac linear distributions created by the accumulation of charges on edges of the mesh. We refer to [4] for entire details.

The range of $\widetilde{Y}^+$ being not included in the approximation space X_h, it is natural to seek for an approximation of $\widetilde{Y}^+$ of the form $\widetilde{\mathscr{Y}}_h^+ = P^h\widetilde{Y}^+$, with P^h a projection on X_h. Since $\widetilde{Y}^+$ can be decomposed into the sum of two operators $-\mathrm{i}k\mathbf{n}\times\sum_p\chi_p G\chi_p$ and $(\mathrm{i}k)^{-1}\mathbf{n}\times\sum_p\chi_p\nabla G\nabla\cdot\chi_p$ of respective order -1 and 1, it is probably more advisable to realize projections adapted to each term. We define the projection $\Pi^h_{\mathrm{edge}}\mathbf{u} = \sum_i f_i(\mathbf{u})\mathbf{e}_i$, where $f_i(\mathbf{u})$ denotes the flux of $\mathbf{u}$ through the edge i (we suppose that the DoF are normalized by their flux). We also denote by Π^h_{node} the projection on the scalar P^1 finite element space S_h defined by $\Pi^h_{\mathrm{node}}u = \sum_i u(x_i)e_i$, with e_i the DoF attached to the node x_i. Noting that $\mathbf{n}\times\nabla(S_h)\subset X_h$, we propose to approach $\widetilde{Y}^+$ on X_h by

$$\widetilde{\mathscr{Y}}_h^+ = \frac{k}{\mathrm{i}}\sum_p \chi_p^h\Pi^h_{\mathrm{edge}}\mathbf{n}\times G\chi_p^h + \frac{1}{\mathrm{i}k}\mathbf{n}\times\sum_p\chi_p^h\nabla\Pi^h_{\mathrm{node}}G\nabla\cdot\chi_p^h. \tag{35}$$

The advantage of this approximation is that it takes into account in an exact way the rotational $\mathbf{n}\times\nabla$, which is essential for the regularization process working in the product $\mathbf{n}\times L\widetilde{Y}^+$.

The matrix $\widetilde{Y}_h^+$ is then simply built as $\widetilde{Y}_h^+ = i_h^{-1}\widetilde{\mathscr{Y}}_h^+ i_h$, with i_h the mapping that associates to a numerical vector $U_h\in\mathbb{C}^{N(h)}$ the current $\mathbf{u}_h = \sum U_h^j\mathbf{e}_j$. Let remark that this matrix is sparse because, thanks to the localizations, only the coefficients (i,j) corresponding to DoF $\mathbf{e}_i$ and $\mathbf{e}_j$ such that the edges i and j are located on the same patch U_p, do not vanish.

We generate the cut-off functions χ_p by tensorization in three dimensions of a quadratic partition of the unity of the real axis which is restricted on Γ [5]. In order to satisfy (31), the volumic partition is furthermore adapted to the frequency by dilating a fixed partition $(\tilde{\chi}_p)$ using $\tilde{\chi}_p(kx)$. In terms of $\tilde{\chi}_p$, (31) simply becomes $\max_{x\in\mathbb{R}^3}(\sum_p|\nabla\tilde{\chi}_p|^2)<2$.

This scheme may appear expensive, since we have to perform first the product with the sparse matrix $\widetilde{Y}_h^+$, and then two independent products involving $[\mathbf{n}\times L]_h$ and $[\mathbf{n}\times K]_h$. However, a judicious discretization of these two operators by FMM, enables us to gather the transfer and reconstruction phases for L and K. Therefore, the additional cost of this scheme compared with a classical equation is the cost of two sparse products, one involving $\widetilde{Y}_h^+$ and the other a near FMM matrix, which is very competitive.

5. Numerical results

The scattering objects we consider are shown in Fig. 1. The cone-sphere (of length 66 cm) and the Channel cavity (of length 1.36 m) are, respectively, described in [4,2]. The discretization of these geometries complies with the criterion of around six edges per wavelength.

The iterative solver is GCR [13].

Before performance results, let us present some phenomenological results.

Fig. 2 gives bistatic radar cross section (RCS) diagrams calculated with GCSIE and CFIE equations. It shows that the precision of the GCSIE is comparable to the one of the CFIE for RCS applications. Quality of the GCSIE implementation can be also estimated by comparing the solution of the equation with the right-hand side. Indeed, these quantities are the same in the ideal case where $\widetilde{Y}^+ = Y^+$. Fig. 3 confirms that they are very close and that the model

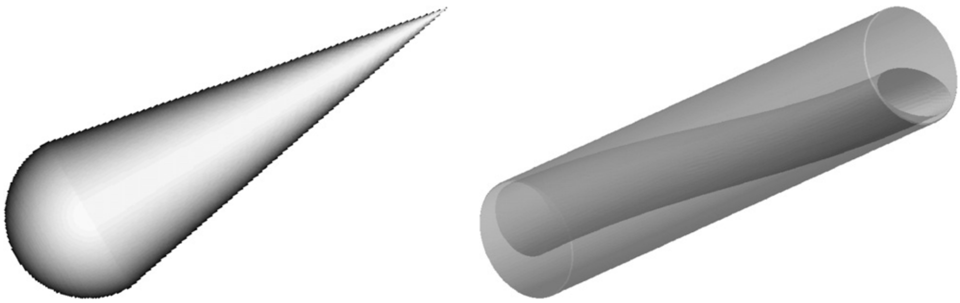

Fig. 1. Geometries of the cone-sphere and the Channel cavity.

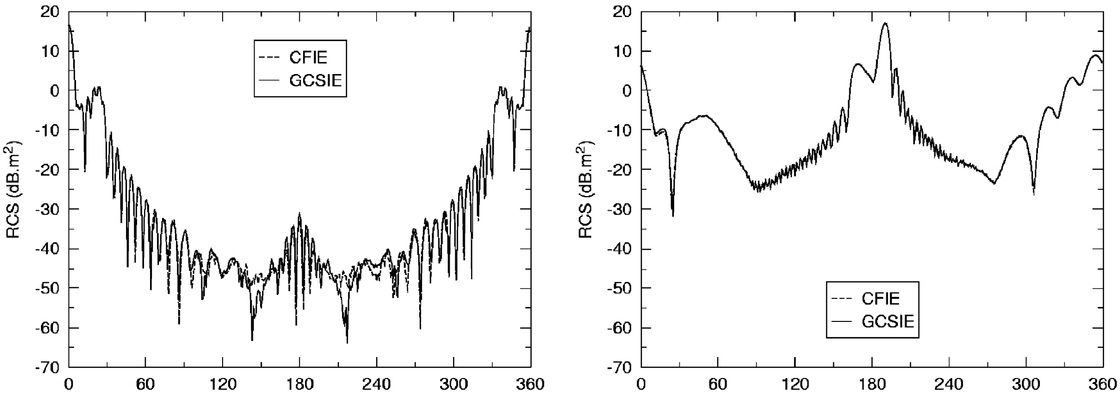

Fig. 2. Bistatic RCS of the cone-sphere at 27 GHz meshed with 153033 DoF (left) and of the Channel cavity at 5 GHz meshed with 258384 DoF (right).

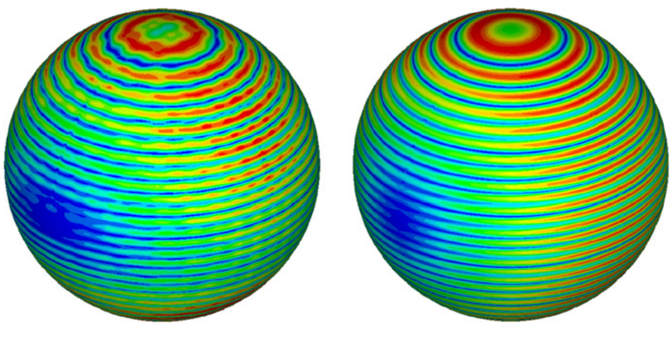

Fig. 3. Real part of the solution of the GCSIE (left) and the right hand side (right).

(16) is relevant at high frequencies. For a constant quadratic partition of the unity, the condition (31) implies that, for a sufficiently high frequency, the equation GCSIE is well-posed, and that, a contrario, the GCSIE may have a bad behavior at low frequency. On a sphere (of radius 1 m), when the frequency decreases, we observe an explosion of the number of iterations required to attain a residue in 10^{-4} (Fig. 4, left). But, if we adapt the cut-off functions χ_p to the frequency in such a way that the condition (31) is verified (by taking larger supports), then we stabilize the iterative convergence.

Numerical experiments also reveal that the GCSIE is stable with the mesh thinness. For the sphere at 0.6 GHz, Fig. 4 (right) shows that the number of iterations remains stable when the mesh becomes thinner and thinner.

We now compare the numerical performances of the GCSIE with other classical equations.

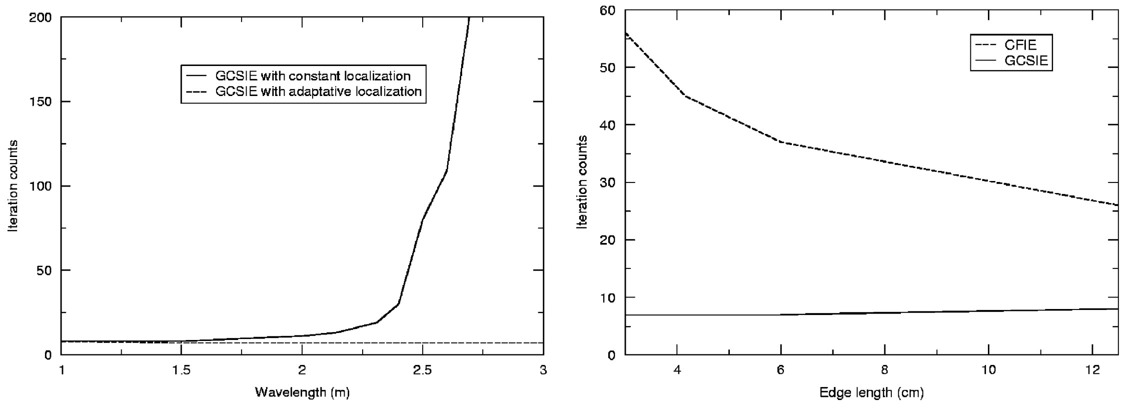

Fig. 4. Iteration counts for the sphere, in function of the wavelength (left), and in function of the mesh thinness (right).

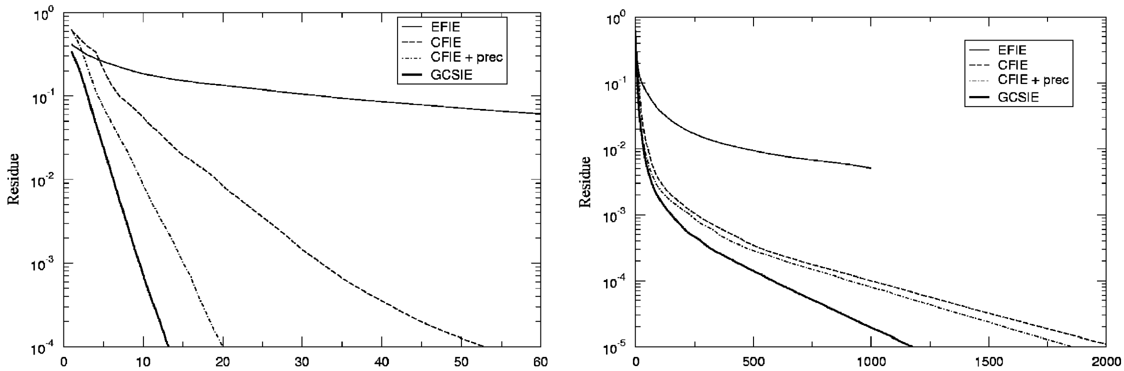

Fig. 5. Convergence curves for the cone-sphere at 27 GHz meshed with 258384 DoF (left), and for the Channel cavity at 7 GHz meshed with 309711 DoF (right).

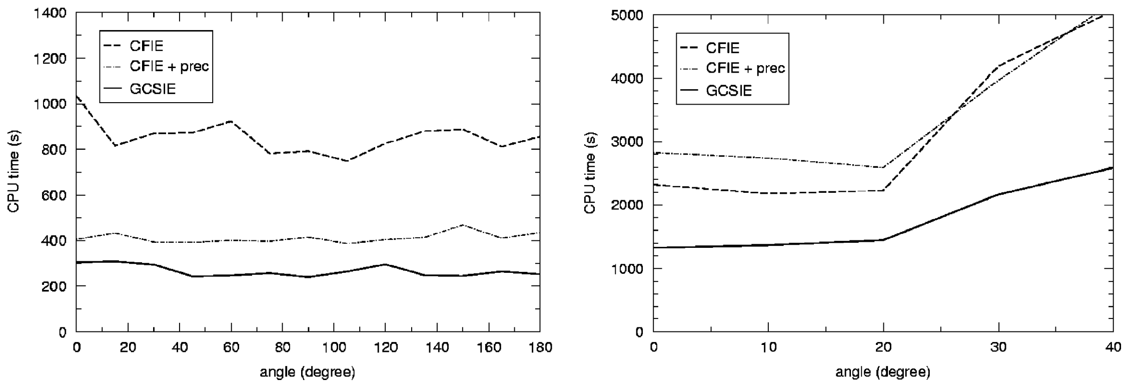

Fig. 6. Resolution time for the cone-sphere at 27 GHz meshed with 258384 DoF (left), and for the Channel cavity at 7 GHz meshed with 309711 DoF (right).

Fig. 5 shows that the GCSIE speeds up the iterative convergence compared with the EFIE, but also, which is less trivial, compared with the CFIE and even with the CFIE preconditioned by an approximated inverse. Contrary to this preconditioner, efficient for convex obstacles, but whose behavior is uncertain for cavities, the GCSIE seems to be uniformly efficient.

With regard to the time for solving the linear system, we note almost the same profits than those observed for the convergence speed. Results presented in Fig. 6 correspond to calculus realized on eight nodes of a parallel machine, where each node is equipped with four processors (1.25 GHz) and have a memory of 4 Gb. Residue was fixed to 10^{-4} for the cone-sphere and to 10^{-3} for the Channel cavity. Resolution times depend on the incidence angle of the plane wave excitation. Nevertheless, we can conclude that the resolution time for the GCSIE is about twice smaller than the one obtained with the most efficient classical equation, CFIE with preconditioner for the cone-sphere, CFIE without preconditioner for the Channel cavity.

6. Conclusion

We have proposed in this paper a new formulation particularly suitable for the simulation of the scattering problem at high frequencies. The integral formulation is shown to be well-posed and produces a well-conditioned linear system that can be solved by means of standard iterative methods without the need of any preconditioner. At the end, numerous experiments show the reliability of the method. In particular, a few problems known to be difficult are solved within a very few iterations.

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